On the concept of Approximate Cofibration

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***Abstract*— In this article we study an important concept in the theory of fibration and cofibration, namely approximate cofibration (A-cofibration), which is the dual of the concept of approximate fibration [5, 10, 13], we give some examples. Following the known problems concerning the concept of cofibration as; the composition, the product, the pullback, the relation with retracts and so on, [1, 4, 6, 7 and 13], we give some similar results concerning A-cofibration.**

***Subject Classification*: 55M20; 55R10**

***Index Terms*—Fibration, cofibration**

I. INTRODUCTION

There are two forms for fibration and cofibration as "Lifting Problem and Extension Problem, the familiar "Homotopy Extension Property" is special case of Extension Problem, in [5] give the formula of approximate Homotopy Lifting Property (A-HLP), which are generalize the concept of fibration, and hence holds for the larger set of maps. Poul and Matthey established a general method to produce co-fibrant approximations in the model category. In this work we study the some properties of approximate cofibration (A-cofibration) concept.

The word of mapping means continuous function, the word of space means topological space, and we replaced a long word (neighborhood) by abbreviation (nbd).

II. PRELIMINARIES AND DEFINITIONS

Firstly, we will begin with the following terminology and notations [2, 9];

Definition(1-1): Let $f, g: E \to B$ be mapping and ξ be an open cover of B ,we say that f,g are - ξ closed iff given $e \in E$ then there exist $w \in \xi$ such that $f(e), g(e) \in w$.

Definition(1-2): A map $p: E \to B$ have to approximate lowering homotopy property (A-LHP) w.r.t X iff given a map $h: B \to X$ and a homotopy $f_t: E \to X$ such that $hop = f_0$ and open cover ξ of X, then there exist a homotopy $h_t: B \to X$ with $h_0 = h$ and $h_t op\ f_t$ are ξ -closed in f_t , for all $.t \in I$ Now let $\mathfrak{R}$ be a given class of topological space , a map p is a cofibration w.r.t $\mathfrak{R}$ iff has $p: E \to B$ (LHP) w.r.t each $X \in \mathfrak{R}$.

Definition (1-3):

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1- Let X_1, X_2, Y be three topological spaces , let ,$X = \{X_1, X_2\}, f = \{f_1, f_2\}$, where $f_1: X_1 \to Y$, $f_2: X_2 \to Y$ are two fiber space and $\alpha: X_2 \to X_1$ such that $f_1 o\alpha = f_2$ then $\{X, f, Y, \alpha\}$ is a M-fiber space (Mixed fiber space), If $X_1 = X_2 = X, \alpha = identity$,$f = f_1, f_2$ then $\{X, F, Y\}$ is the usual fiber space.

2- Let $\{X, f, Y, \alpha\}$ be a M-fiber space let $y_0 \in Y$ then $F = \{f(y_o)\}$ is the M-fiber over y_o.

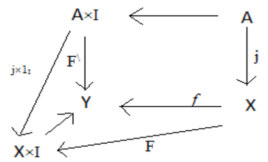
Definition(1-4) : Two maps f, κ: X →Y, are said to be U-close, U∈cov(Y), provided that for each x∈X, one element of U containing both f (x) and κ (x).

A map f: X→Y, is a near-homeomorphism if for any U∈cov(Y), there exist a homeomorphism of X onto Y which is U-close to f.

Next, a maps f, κ: X→Y, are U-homotopy, U∈cov(Y), iff it is a homotopic by a homotopy H: X×I→Y, and H({x}×I) contained in one element of *U*.

Definition (1-5): A proper map P: E→B, between locally compact ANR's, has the approximate homotopy lifting property (A-HLP), w.r.t. a space X, provided that , given a U∈cov(B), a maps κ: X→E, and H: X×I→B, such that P∘ κ= H_o, there exist G: X×I→E, such that G_o= κ, and P∘ G is U-close to H; a map G is called U-lift of H. Maps with the A-HLP, w.r.t. all spaces is A-fibration [5].

Definition (1-6): A space X is said to be an ANR's (absolute nbd retract), if for any space Y in which X can be embedded as a closed set there exist a *nbd* V of X in Y such that X is a retract of V, (i.e. there exist r: V→X, such that r ∘ j = $\mathbf{1}_X$.



Next, a map j: A→X is said to be have a homotopy extension property (HEP) w. r. t. a space Y, Provided that given a map f: X→Y, a homotopy F\: A×I→Y, of f\: A→Y such that f∘ j= $F^{\backslash}_o$, Then there exist a homotopy F: X×I →Y, Such that the shown diagram, commutes, See [1, 6, 7, 10].

Definition (1-7): A map j: A→X, is called a cofibration iff it has the (HEP) w.r.t. all spaces. Fibration and co-fibration, as well as various modifications of this notion, have some nice properties, which make them useful in studying both spaces and maps, There are several authors showed that some other classes of maps, e.g. cell-like maps, enjoy similar lifting properties [12]. This motivated Coram and Duvall [5], to define the notion of approximate fibrations.

This is a map P: E→B, between say compact metric ANR's (absolute nbd retracts), provided given ε>***o***, there exist δ>***o*** such that for each space X, whenever h: X→E and H: X×I→B are maps with H_o is δ-close to P ∘ h there exists Ĥ: X×I→E such that Ĥ (x, o) =h (x) and d(H(x, t), P∘ Ĥ (x,

$t)) < \varepsilon$, for each $x \in X$ and $t \in I$ "i.e. the given diagram commutes only up to ε".
We begin with survey about some of concerning the concept is cofibration;
Definition(1-8): Let j: X⟶Y, be a map, define cyl(X) = I×X, and let i_1: X⟶ cyl (X), be the map $i_1(x) = (0, x)$, Let
$cyl(j) = (I \times X)\bigcup_X Y,$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & \mathrm{Cyl}(X) \arrow[d] \\ Y \arrow[r, "i"] & \mathrm{Cyl}(j) \end{tikzcd}$$

There is an evident map $k : \{(I \times X)\bigcup_X Y\} \to I \times Y,$ which map I×X by $\mathbf{1}\times j$, and Y by i. And j is cofibration iff there a map $r : I \times Y \to [(I \times X)\bigcup_X Y])$, where $r \circ k = 1$, this means I×Y can be pushed down continuously onto subspace $\{(I \times X)\}$.
In other hands, consider f_o: X⟶Y, is given, and a subspace A⊂ X, one is also given homotopy f_t: A⟶Y, of $f_o \mid A$, which one would like to extend to homotopy f_t: X⟶Y, of f_o, if pair (X, A) is such that this extension problem can always be solved, one says that (X, A) has HEP. Thus (X, A) has HEP if all maps (X×0 ∪ A×I) ⟶Y can be extended to map X×I⟶Y, in particular, the HEP for (X, A) implies that the identity map (X×0 ∪ A×I)⟶ (X×0 ∪ A×I) extends to X×I⟶ (X×0 ∪ A×I) so (X×0 ∪ A×I) is retract of $X \times I$.
Conversely there is a retraction X×I⟶ (X×0 ∪ A×I), then by composing with this retraction we can extend every map (X×0 ∪ A×I) ⟶Y to a map X×I⟶Y. Thus the (HEP) for (X, A) is equivalent to (X×0 ∪ A×I) being retract of X×I. This relation criterion allows one to give simple examples of pairs (X, A) which do not have the (HEP), such as (I, (0,1)), since **r**: I×I⟶ [I×0 ∪ (0,1)×I] would have the compact image.
A slightly less trivial example, [13], is j: S⟶I, where S is the sequence {1/n}, together with its limit point o, it is not hard to show that there is no retraction.
A quite useful result in the positive direction is:

Proposition(1-9): [8] If (Y, X) is a CW pair, then [(I×X) $\cup_X$ Y], is retract of I×Y, hence [(I×X) $\cup_X$ Y], is a cofibration.
Proposition(1-10): [7] If (X, A) satisfies the (HEP), and A is contractible, then the quotient map q: X⟶X | A, is homotopy equivalence.

Proposition(1-11): [8] If (X, A) and (Y, A) are the (HEP), and f: X⟶Y, is a homotopy equivalence with $f \mid A = \mathbf{1}$. Then f is a homotopy equivalence rel A.

Finally we find it easier to avoid declaring upfront, which it is more convenient to define fibration or cofibration.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j"'] & Y \arrow[d, "p"] \\ X \arrow[r, "k"] & B \end{tikzcd}$$

An ordered pair of classes of map j: A⟶X, p: Y⟶B, has relative lifting property (RLP) [8, 11], if for any diagram, a filler f: X⟶Y, exist.
Definition(1-12) [11]: Given a class ∑ of maps; we call j: A⟶X, a ∑-cofibration if (j, p) has the (**RLP**) for all p∈∑.

And p: Y⟶B, is called a ∑-fibration if (j, p) has the (**RLP**) for all j∈∑.
Definition(1-13): A proper map j: X⟶Y, between locally compact ANR's has the Approximate homotopy extension property (A-HEP) w.r.t. a space Z, provided that given U∈cov (Z), a map f: Y⟶Z, a homotopy h_t: X⟶Z, such that $f \circ j = h_o$, there exist a homotopy f_t: Y⟶Z, such that $f_o = f$, and $f_t \circ j$ is U-close to h_t, where f_t is said to be U-extended of h_t,

$$\begin{tikzcd} X \arrow[d, "j"'] \arrow[dr, "h_t"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

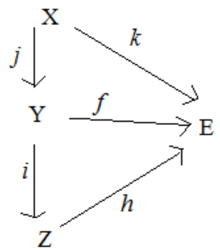
A map with (A-HEP) w. r. t. all spaces are called A-cofibration.

The above definition of course generalizes the usual cofibration (HEP), thus the A-cofibration (A-HEP) holds for a larger set of maps. As in [3, 5, 20], we called (Y, X) an A-cofibered pair. It is clearly that, a cofibration is an A-cofibration, also the near-homeomorphism is A-cofibration.

Proposition(1-14): Let j_1 : X⟶Y is cofibration and j_2 : Y⟶Z is A-cofibration, then $j_2 \circ j_1$ is A-cofibration.
Proof: Let $f^{\backslash}_t$: X⟶E and h: Z⟶E, be a given such that $h \circ j_2 \circ j_1 = f^{\backslash}_o$, since j_1 is cof n, then there exist f, f_t: Y⟶E, such that $f_o = f$ and $f_t \circ$ also j_2 is A-cofibration, then there exist h_t: Z⟶E, such that $h_t \circ j_2$ is U-close to f_t (U∈cov (E)), and $h_o = h$.

$$\begin{tikzcd} X \arrow[d, "j_1"'] \arrow[ddr, "f^{\backslash}"] & \\ Y \arrow[dr, "f"] \arrow[d, "j_2"'] & \\ Z \arrow[r, "h"] & E \end{tikzcd}$$

Hence we have that $h_t \circ (j_2 \circ j_1)$ is U-close to $f^{\backslash}_t$, then $(j_2 \circ j_1)$ is A-cofibration.
Corollary(1-15): If j: X⟶Y and i: Y⟶Z, be a maps such that j is cofibration and $i \circ j$ is A-cofibration, then i is A-cofibration.
Proof: Given a maps f_t: Y⟶E and h: Z⟶E, such that $h \circ i = f_o$, since $(i \circ j)$ is A-cofibration then for any a given κ_t: X⟶E, such that $h \circ i \circ j = \kappa_o$, there exist h_t: Z⟶E, such that $h_t \circ i \circ j$ is U-close to κ_t "U∈cov(E)", and $h_o = h$, and since j is cofibration then there exist f , f_t: Y⟶E, Such that $f = f_o$ and $f_t \circ j = \kappa_t$. Hence we have $h_t \circ i$ is U-close to f_t, therefore i is A-cofibration.

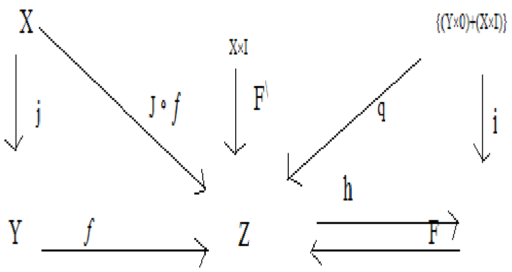


III-Maine results of A-cofibration related with A-retract

Firstly we will give the following terminology and notation; also we will introduce some definitions that we need.

If X is a subspace of a space Y such that the inclusion map $X\subset Y$, is an A-cofibration, then the pair (Y, X) is called an A-cofibered pair or is said to possess the (A-HEP). A condition for (Y, X) to be an A-cofibered pair is the existence of approximate retraction (A-retracts), $\mathbf{r}_A$: $Y\times I \rightarrow \{(Y\times 0) \cup (X\times I)\}$.

Definition(2-1): The inclusion map j: $X\rightarrow Y$ is called an approximate retract (A-retract), iff $\mathbf{r}_A \circ j$ is U-close to $\mathbf{1}_X$ for any $U\in cov(X)$. If $j \circ \mathbf{r}_A$ is U-homotopic to $\mathbf{1}_Y$ $U\in cov(Y)$, then j is an approximate deformation retracts (A-D**r**). If $j \circ \mathbf{r}_A \simeq_U \mathbf{1}_Y$ rel X, then j is an (A-SD**r**).



The first two theorems provide a tool for constructing examples of maps, which is the A-cofibration.

Theorem(2-2): If j: $X\rightarrow Y$ is an A-cofibration then j is a near-homeomorphism.

Proof: Conceder the following diagram;

Let j: $X\rightarrow Y$ be an A-cofibration, Consider $Z= [(Y\times 0) +_f (X\times I)]$, is the quotient space of topological sum obtained by identifying (x, 0) with (j(x), 0).

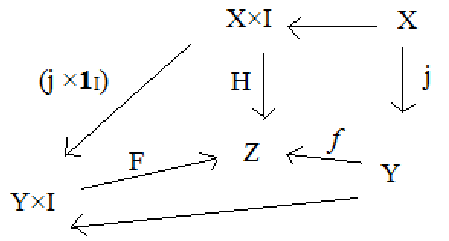
Let q be the quotient map q: $[(Y\times 0) + (X\times I)]\rightarrow Z$, that there is a map h: $Z\rightarrow Y\times I$, define as $h \circ q\,(y, 0) = (y, 0)$, $y\in Y$ and $h\circ q\,(x, t)= (j(x), t)$.

Let f: $Y\rightarrow Z$ and $F^{\backslash}$: $X\times I\rightarrow Z$, such that $f(y) = q\,(y, 0)$ and $F^{\backslash}(x, t) = q\,(x, t)$, Since j is an A-cofibration, then for any $U\in cov(Z)$, there exist F: $Y\times I\rightarrow Z$, such that $F(y, 0) = q\,(y, 0)$ and $F(j(x), t)$ is U-close to $q\,(x, t)$;

Hence $F \circ h$ is U-close to $\mathbf{1}_Z$. Then h is near-homeomorphism of Z to $h(Z) = \{(Y\times 0) +_f (j(X)\times I)\}$; Also $q \mid X\times I$, is homeomorphism of $X\times I$ onto $q\,(X\times I)$, And consequently; $H \circ q \mid X\times I$, will be the near-homeomorphism of $(X\times I)$, to $(h \circ q\,(X\times I) = j(X)\times I)$.

Theorem(2-3): The pair (Y, X) is A-cofibration iff $[(Y\times 0) \cup (X\times I)]$ is A-retract of $Y\times I$.

Proof: Let f: $Y\rightarrow Z$, h: $X\rightarrow Z$ and H: $X\times I\rightarrow Z$, such that $H_o = f|X$, Define $F = \{f, H\} \circ \mathbf{r}_A$: $Y\times I\rightarrow Z$, since for any $U\in cov(Z)$ $\mathbf{r}_A \circ j$ is U-close to $\mathbf{1}_{(Y\times 0)\cup(X\times I)}$, this means that for any map from $[(Y\times 0) \cup (X\times I)]$ to Z, has an U-extention from $Y\times I$ to Z, it follows that $F\circ (j\times \mathbf{1}_I)$ is U-close to H, and $F_0 = f$, hence (Y, X) is an A-cofibered pair,



III. Conversely:

If (Y, X) is an A-cofibered pair, let $Z = \{(Y\times 0) \cup (X\times I)\}$, and $\{f, H\}$ be the identity map, then an U-extension of H starting at f is A-retraction $\mathbf{r}_A$: $Y\times I \rightarrow [(Y\times 0) \cup (X\times I)]$.

The condition that X be closed is not very restrictive; also not all A-cofibration are closed however the most trivial example of a non-closed A-cofibration is the pair (Y, x) where Y is the two-point space $\{x,\ y\}$ with the trivial topology.

If X is a subspace of a space Y, the mapping cylinder of the inclusion $X\subset Y$, may be identified with the subset $\{Y\times 0 \cup X\times I\}$ of $Y\times I$, also if $\{Y\times 0 \cup X\times I\}$, is A-retract of $Y\times I$, then the subspace topology inherited from $Y\times I$ is identical with the mapping cylinder topology, these topologies are also identical if X is closed, even if no A-retraction of $Y\times I$ to $\{Y\times 0 \cup X\times I\}$ exist, hence they need not be identical for any pairs (Y, X).

Corollary(2-4): If (Y, X) is an A-cofibered pair, so $\{(Y\times 0) \cup (X\times I)\}$ is (A-SD**r**), of $Y\times I$.

Proof: The U-homotopy between $j \circ \mathbf{r}_A$ and $\mathbf{1}_{Y\times I}$ will be given as;

$H_U(y, t, t^{\backslash})= \{(P_1 \circ \mathbf{r}_A\,(y, (1 - t^{\backslash})\, t\,)\}, \{(1 - t^{\backslash}\,)P_2 \circ \mathbf{r}_A\,(y, t) + t^{\backslash} t)\}$, which P_1, P_2 is the projection on X and I respectively.

Lemma(2-5): The pair (Y, X) is an A-cofibered pair iff there exists a map φ: $Y\rightarrow I$, such that $X\subset \varphi^{-1}(0)$, and U-homotopy H_U: $Y\times I\rightarrow Y$, such that $H_U(y, 0) = y$,

$H_U(x, t) = x$, and $H_U(y, t) \in X$ whenever $t > \varphi(y)$.

Proof: Suppos j is A-cofibration, there exists A-retract $\mathbf{r}_A$: $Y\times I \rightarrow \{(Y\times 0) \cup (X\times I)\}$, then φ and H_U are define as follows; $\varphi(y) = \sup\{t - P_2 \circ \mathbf{r}_A\,(y, t)\}$ and $H_U(y, t) = P_1\circ \mathbf{r}_A\,(y, t)$, $y\in Y$, $t\in I$.

Conversely: If φ and H_U are exists, then $\mathbf{r}_A$ is defined by

$$\mathbf{r}_A(y,t)=\begin{cases} H_U((y,t),0) & t\le\varphi(y) \\ H_U((y,t),(y,t-\varphi(y))) & t\ge\varphi(y)\end{cases}$$

Remark:

If $\varphi(y)<1$, then $H_U(y, \varphi(y))\in \overline{H_U(y\times(\varphi(y),1))} \subset \overline{X}$,

thus replacing $H_U(y, t)$ by $\overline{H}_U(y,t\wedge\varphi(y))$, We have the following result.

Corollary(2-6): If (Y, X) is A-cofibration, so is (Y, $\overline{X}$).

The following lemma is generalized of (1-7), in [3], and we needed in the last section, that we well give the proof of it.

Lemma(2-7): If subspace X of space Y is (A-D**r**) of Y, then the inclusion map j: $X\rightarrow Y$ is a U-homotopy equivalence.

Proof:
Since X is (A-D**r**) of Y, then there exist D_A : Y×I⟶Y, such that d_1 is A-retract of Y onto X ($\mathbf{r}_A$: Y⟶X), then $\mathbf{r}_A \circ j$ is the identety on X; then j is a homotopy equivalence.

Lemma(2-8): Suppose that P: E⟶B is A-fibration, with X is an (A-SD**r**) of Y, and that there exist a map φ: Y⟶I, such that $X = \varphi^{-1}(0)$, then a U-commutative diagram;

$$\begin{tikzcd} X \arrow[r, "f^{\backslash\backslash}"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f^{\backslash}"'] & B \end{tikzcd}$$

My be filled in with a map *f*: Y⟶E, such that P∘ *f* is U-close to $f^{\backslash}$, and $f \circ j = f^{\backslash\backslash}$, *f* is unique up to a U-homotopy; (U∈cov(B)).

Proof: By hypothesis, there exist (A-SD**r**), D_A: $j \circ \mathbf{r} \simeq_U \mathbf{1}_Y$, rel X. Define $\check{D}_A$: Y×I⟶Y, by;

$$\check{D}_A(y, t) = \begin{cases} D_A(y, t/\varphi(y)), & t<\varphi(y) \\ D_A(y, 1), & t \geq \varphi(y). \end{cases}$$

Since P is A-fibration, there exist U-homotopy, $F^{\backslash}_U$: Y×I⟶E, such that P∘ $F^{\backslash}_U$ is U-close to $f^{\backslash} \circ \check{D}_A$, and $F^{\backslash}_U(y, 0) = f^{\backslash\backslash} \circ \mathbf{r}$; we defined *f* as $f(y) = F^{\backslash}_U(y, \varphi(y))$. If *f*: Y⟶E, is such that $f \circ j = f^{\backslash\backslash}$, then $f \simeq_U f \circ j \circ \mathbf{r} = f^{\backslash\backslash} \circ \mathbf{r}$, rel X.

Theorem(2-9): Suppose that P: E⟶B is A-fibration and j: X⟶Y is A-cofibration, which X is closed, then any U-commutative diagram;

$$\begin{tikzcd} {[(Y\times 0)\cup(X\times I)]} \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ Y\times I \arrow[r, "F"'] & B \end{tikzcd}$$

May be filled in with a homotopy $F^{\backslash}$: Y×I⟶E, such that P ∘ $F^{\backslash}$ is U-close to F, for any U∈cov(B), and $F^{\backslash} \mid \{(Y\times 0) \cup (X\times I)\} = f$.

Proof: By corollary (4-3), [(Y×0) ∪ (X×I)] is (A-SD**r**) of Y×I; That is D_A: $j \circ \mathbf{r} \simeq_U \mathbf{1}_{Y\times I}$, rel [(Y×0) ∪ (X×I)]. And by lemma (5-3), there exist a function ψ: Y⟶I, such that $X = \psi^{-1}(0)$. Define φ: Y×I⟶I, by φ $(y, t) = t\psi(y)$, then $[(Y\times 0) \cup (X\times I)] = \varphi^{-1}(0)$.
And hence the theorem follows from (1-4).

Definition(2-10): Let j: X⟶Y and P: E⟶B are maps, a map pair $f = (f^{\backslash\backslash}, f^{\backslash})$: j⟶P, is a pair of maps $f^{\backslash\backslash}$: X⟶E and $f^{\backslash}$: Y⟶B, such that the diagram is U-commutes, U∈cov(B).

$$\begin{tikzcd} X \arrow[r, "f^{\backslash\backslash}"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f^{\backslash}"'] & B \end{tikzcd}$$

And a map ġ: Y⟶E defines a map pair Q(ġ) =(ġ ∘ j, P ∘ ġ): j ⟶P, ġ is called a U-lifting of the pair Q(ġ).

Theorem(2-11): Let j:X⟶Y, be map with j(X) closed; then j is A-cofibration and U-homotopy equivalence, iff a map pair *f*: j⟶P, with P: E⟶B is A-fibration, has U-lifting.

Proof: Suppose that j is A-cofibration and since P be a given A-fibration then the first direction is just lemma (1-4).

Conversely; since $л_0$: Z^I⟶Z is A-fibration for any space Z; hence we consider the U-commutative diagram;

$$\begin{tikzcd} X \arrow[r, "G"] \arrow[d, "j"'] & Z^I \arrow[d, "л_0"] \\ Y \arrow[ur, "F"] \arrow[r, "\dot{g}"'] & Z \end{tikzcd}$$

Where F is U-lifting of its map pair. And so j: X⟶Y must be A-cofibration. Next, we may assume j is an inclusion map, and since X⟶{p}, is A-fibration ({p} denotes a one-point space); then an A-retraction $\mathbf{r}_A$: Y⟶X is obtained as U-lifting of the map pair of the following diagram;

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & X \arrow[d] \\ Y \arrow[r] & \{p\} \end{tikzcd}$$

The map P: Y^I⟶Y×Y, defined by P(*w*) =(*w*(0), *w*(1)) is also A-fibration[8], and hence the map pair of the following diagram;

$$\begin{tikzcd} X \arrow[r, "f^{\backslash\backslash}"] \arrow[d, "j"'] & Y^I \arrow[d, "p"] \\ Y \arrow[r, "f^{\backslash}"'] & Y\times Y \end{tikzcd}$$

With $f^{\backslash\backslash}(x)(t) = x$, $f^{\backslash}(y) = (y, \mathbf{r}(y))$, has U-lifting ġ: Y⟶$Y^I$ associate to (A-SD**r**) of Y to X, and the complete of proof follows from (3-7).

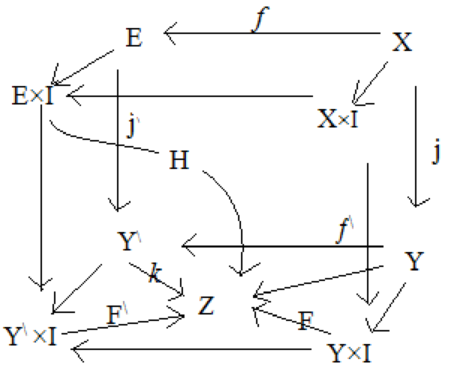
Corollary(2-12): In the above theorem if the equivalently holds, then the U-lifting of *f* is unique up to a U-homotopy relative to j(X).

The proof of the following theorem is a similar fashion;

Theorem(2-13): For a map P: E⟶B, the map pair *f*: j⟶P, with j is a closed A-cofibration, has U-lifting iff P is an A-fibration and U-homotopy equivalence.

IV. Induced A-cofibration:

Let j: X⟶Y, *f*: X⟶E, be a maps and $Y^{\backslash} = E \; {}_j\cup_f \; Y$, be the cofibers sum of Y and E, which is the set of all equivalents classes of topological sum under the equivalence relation generated by [$e \sim y \Leftrightarrow \exists\, x \in X$: $e = f(x), y = j(x)$]. Let **q:** E+Y⟶ $E \; {}_j\cup_f \; Y$ is the identification map.



Define $f^{\backslash}$: Y→Y$^{\backslash}$, j$^{\backslash}$: E→Y$^{\backslash}$, as the composition of **q** with the inclusions of E and Y into E+Y res., then j$^{\backslash}$ is called the pushout of j by f.

Theorem(3-1): The pushout of an A-cofibration is also A-cofibration.

Proof: Let j: X→Y be an A-cofibration, and f: X→E be a map. Let j$^{\backslash}$: E→Y$^{\backslash}$ be the pushout of j by f; so for any space Z, let κ: Y$^{\backslash}$→Z and H: E×I→Z, such that κ ∘ j$^{\backslash}$ = H_o, and we have that κ∘ $f^{\backslash}$: Y→Z and H(f×$\mathbf{1}_I$): X×I→Z, such that (κ ∘ $f^{\backslash}$) ∘ j= H_o (f×$\mathbf{1}_I$); since j is an A-cofibration, there exist F: Y×I→Z, such that κ ∘ $f^{\backslash}$ = F_o, and F(j×$\mathbf{1}_I$) is U-close to H(f×$\mathbf{1}_I$), which U∊cov (Z). Hence define F$^{\backslash}$: Y$^{\backslash}$×I→Z, by F$^{\backslash}$ ($f^{\backslash}$(y), t)= F(y, t), and F$^{\backslash}$ (j$^{\backslash}$ (y), t)= H(e, t),
then F$^{\backslash}$ (j$^{\backslash}$ ×$\mathbf{1}_I$) is U-close to H, and F$^{\backslash}$ ($f^{\backslash}$ (y), 0) = F(y,0)= κ ∘ $f^{\backslash}$ (y) = κ(**q**(y)), also F$^{\backslash}$ (j$^{\backslash}$(y), 0)= H(e, 0) = κ ∘ j$^{\backslash}$ (e)= κ(**q**(e)), which (**q:** E+Y→$E_j \cup_f$ Y).

Theorem(3-2): If j: X→Y and i: X$^{\backslash}$ →Y$^{\backslash}$, are A-cofibration with X closed in Y, then [(Y, X) × (Y$^{\backslash}$, X$^{\backslash}$)] = (Y×Y$^{\backslash}$, Y×X$^{\backslash}$ ∪ X×Y$^{\backslash}$) is also A-cofibration.

Proof: Let φ: Y→I and H_U: Y×I→Y, be as described in lemma (2-3),

Let ψ and G_U, be the corresponding maps for (Y$^{\backslash}$, X$^{\backslash}$); define η: Y×Y$^{\backslash}$ →I and F_U: Y×Y$^{\backslash}$ ×I→Y×Y$^{\backslash}$, by F_U (y, $y^{\backslash}$, t) = [H_U (y, t ∧ ψ($y^{\backslash}$)), G_U($y^{\backslash}$, t ∧ φ(y))] and η(y, $y^{\backslash}$)= [φ(y) ∧ ψ($y^{\backslash}$)]. Then F_U (y, $y^{\backslash}$, t) = (y, $y^{\backslash}$) and [Y×X$^{\backslash}$ ∪X×Y$^{\backslash}$]⊂ η^{-1}(0) if t=0 or (y, $y^{\backslash}$)∊[Y×X$^{\backslash}$ ∪ X×Y$^{\backslash}$]. Since X is closed then H_U(y, φ(y))∊X whenever φ(y) < 1, Now suppose t > η(y, $y^{\backslash}$); Then either:
φ(y) ≤ ψ($y^{\backslash}$) and φ(y) < t, in which [t ∧ ψ($y^{\backslash}$)] ≥ φ(y) and F_U (y, $y^{\backslash}$, t) ∊ X×Y$^{\backslash}$,
or ψ($y^{\backslash}$) <φ(y) and ψ($y^{\backslash}$) <t, so that [t ∧ φ(y)]>ψ($y^{\backslash}$) and F_U (y, $y^{\backslash}$, t)∊Y×X$^{\backslash}$. This shows that F_U (y, $y^{\backslash}$, t) ∊ (Y×X$^{\backslash}$ ∪X×Y$^{\backslash}$), whenever t > η (y, $y^{\backslash}$), and therefore from lemma (5-3), that (Y×Y$^{\backslash}$, Y×X$^{\backslash}$ ∪X×Y$^{\backslash}$) is A-cofibration.

Theorem(3-3): Suppose that X⊂Y, that there exists a continuous function φ: Y→I, with X⊂φ^{-1}(0) and that there exist a point y_o∊Y \ X, such that φ(y_0) ≠0; also if (Y$^{\backslash}$, X$^{\backslash}$) is a pair such that (Y×Y$^{\backslash}$, Y×X$^{\backslash}$ ∪ X×Y$^{\backslash}$), is A-cofibration, then we have that (Y$^{\backslash}$, X$^{\backslash}$) it self is A-cofibration.

Proof: Let η: Y×Y$^{\backslash}$ →I and F_U: Y×Y$^{\backslash}$ ×I→Y×Y$^{\backslash}$, be as described in (3-2),
We may obviously assume that φ(y_o)=1. Define G_U: Y$^{\backslash}$ ×I→Y$^{\backslash}$ and ψ: Y$^{\backslash}$ →I, by: ψ($y^{\backslash}$)= max{η(y_o, $y^{\backslash}$),1- $\inf_{t\in I}$ φ ∘ P_1 ∘ F_U (y_o, $y^{\backslash}$, t)}and
G_U ($y^{\backslash}$, t) = [P_2 ∘ F_U (y_o, $y^{\backslash}$, t)], which will be satisfy the condition of (2-5).

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